A Note on the Generalized Sum-Capture Problem for Rings

Ashwin Jha

Ruhr-Universität Bochum
Bochum, Germany

letterstoashwin@gmail.com

July 12, 2024

Abstract

The sum-capture problem for abelian groups is generalized over any arbitrary finite ring, for an arbitrary number of sets, and in presence of an arbitrary multiplicative mask.

1 The problem

Let R be a finite ring and fix positive integers $p, q < |R| = N$. Let $A = (A_i)_{1 \le i \le q}$ be a random sequence (or equivalently, an ordered multiset) over R. For any $k \ge 2$, any $\alpha \in R^k$ with at least 2 non-zero coordinates, and any $B_1, B_2, \dots, B_k \subseteq R$, we define

$$\mu_\alpha(A, B_1, B_2, \dots, B_k) = \left| \left\{ (a, b_1, b_2, \dots, b_k) \in A \times B_1 \times B_2 \times \cdots \times B_k : a = \sum_{i=1}^{k} \alpha_i \cdot b_i \right\} \right|.$$

For any p, one can define

$$\mu_\alpha(A; p) = \max_{\substack{B_1, \dots, B_k \subseteq R \\ |B_1| = |B_2| = \cdots = |B_k| = p}} \mu(A, B_1, B_2, \dots, B_k).$$

Note that, $\mu_\alpha(A, B_1, B_2, \dots, B_k)$ is equal to $\dfrac{|A| \times |B_1| \times \cdots \times |B_k|}{|R|}$ in expectation when the sets $A, B_1, \dots, B_k$ are chosen at random. The main problem we consider is to upper bound the deviation of $\mu_\alpha(A; p)$ from qp^k/N that holds with high probability over the random choice of A. For $k = 2$, Babai-Hayes [Bab02, Hay03] (and later Steinberger [Ste13]) proved the following result:

Theorem 1 ([Bab02, Ste13])**.** *Let R be a finite ring, and let $0 \le q \le N/2$. Fix $\alpha = (1, 1)$. For any without replacement sample $A = (A_i)_{1 \le i \le q}$ over R, we have*

$$\Pr\left(\left| \mu_\alpha(A; p) - \frac{qp^2}{N} \right| \ge 4p\sqrt{\ln(N) q} \right) \le \frac{2}{N}.$$

Let $\#\alpha$ denote the number of non-zero coordinates in α. In this short note, for $k \ge 2$, we prove the following two results:

Theorem 2. *Let R be a finite ring, and let $0 \le q \le N/2$. Fix some $\alpha \in R^k$ such that $\#\alpha \ge 2$. For any positive real ϵ and any with replacement sample $A = (A_i)_{1 \le i \le q}$ over R, we have*

$$\Pr\left(\left| \mu_\alpha(A; p) - \frac{qp^k}{N} \right| \ge p^{k-1} \sqrt{2(1+\epsilon) \ln(N) q} \right) \le \frac{4}{N^\epsilon}.$$

Theorem 3. *Let R be a finite ring, and let $0 \le q \le N/2$. Fix some $\alpha \in R^k$ such that $\#\alpha \ge 2$. For any positive real ϵ and any without replacement sample $A = (A_i)_{1\le i\le q}$ over R, we have*

$$\Pr\left(\left|\mu_\alpha(A;p) - \frac{qp^k}{N}\right| \ge 2p^{k-1}\sqrt{2(1+\epsilon)\ln(N)q}\right) \le \frac{2e^2}{N^\epsilon}.$$

Slight simplification: Let $\{i_1, i_2, \dots, i_{\#\alpha}\} \subseteq \{1,2,\dots,k\}$ be the set of non-zero coordinate indices of α. There exists $B_1, B_2, \dots, B_k \subseteq R$ with $|B_i| = p$, such that

$$\begin{aligned}\left|\mu_\alpha(A;p) - \frac{qp^k}{N}\right| = \left|\mu_\alpha(A, B_1, B_2, \dots, B_k) - \frac{qp^k}{N}\right| &= p^{k-\#\alpha}\left|\mu_{\alpha'}(A, B'_{i_1}, B'_{i_2}, \dots, B'_{i_{\#\alpha}}) - \frac{qp^{\#\alpha}}{N}\right| \\ &\le p^{k-\#\alpha}\left|\mu_{\alpha'}(A,p) - \frac{qp^{\#\alpha}}{N}\right|, \end{aligned} \tag{1}$$

where $\alpha' = (1,1,\dots,1) \in R^{\#\alpha}$ and $B'_{i_l} = \alpha_{i_l} \cdot B_{i_l}$. Thus, it is sufficient to study the problem for $\alpha = (1,1,\dots,1)$. Without loss of generality, we assume this form and drop α from the subscript.

It is also clear that the (non-)commutativity of R does not play any role vis a vis the sum-capture problem. Indeed one can define $\mu_\alpha(A;p)$ equivalently using right multiplication.

As a side-effect of the aforementioned simplification *one can completely ignore the multiplicative aspect of R, and simply view it as an additive abelian group of order N.* Henceforth, we simply assume $\#\alpha = k$ as, by virtue of (1), the case of $2 \le \#\alpha \le k-1$ is analogous.

2 A proof

A proof of both the theorems largely extends the Babai-Steinberger approach, delving into basic Fourier analysis, with a brief foray into probabilistic tail inequalities towards the end. We reproduce Steinberger's excellent introductions [Bab02, Ste13] to Fourier analysis (almost verbatim) for the uninitiated, while simultaneously working towards a proof of Theorems 2-3 — the main technical results of this note.

A *character* of R is a homomorphism $\chi : R \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ denotes the multiplicative group of complex numbers. Thus,

$$\chi(x)^N = \chi(Nx) = \chi(0) = 1,$$

which means that the elements in the image of χ are the N^{th} roots of unity, and thus $\chi(-x) = \chi(x)^{-1} = \overline{\chi(x)}$. The *principal character* χ_0 of R is defined as the constant function that maps all $x \in R$ to 1. Thus, $\sum_{x\in R}\chi_0(x) = N$, and for any non-principal character χ and any non-zero $y \in R$,

$$\chi(y)\sum_{x\in R}\chi(x) = \sum_{x\in R}\chi(x+y) = \sum_{x\in R}\chi(x),$$

whence $\sum_{x\in R}\chi(x) = 0$. Then, for distinct characters χ and ξ

$$\sum_{x\in R}\xi(x)\overline{\chi(x)} = 0,$$

follows from the fact that $\xi\overline{\chi}$ is a non-principal character of R.

Let $\hat{R}$ denote the set of characters of R. Then, it is easy to see that $\hat{R}$ forms an abelian group under pointwise multiplication. $\hat{R}$ is called the *dual* group of R, and $R \cong \hat{R}$.

Every function $f : R \to \mathbb{C}$ can be seen as an element of $\mathbb{C}^{|R|}$. This is an N-dimensional space over $\mathbb{C}$. For every $f : R \to \mathbb{C}$, define

$$E_x[f(x)] = \frac{1}{N}\sum_{x\in R} f(x),$$

which gives a natural definition of inner product over $\mathbb{C}^{|R|}$, namely $\langle f, g\rangle = E[f\overline{g}]$. Then, for any $\chi, \xi \in \hat{R}$, we have

$$E[\xi\overline{\chi}] = 0, \qquad \xi \neq \chi$$

More precisely,

$$E[\xi\overline{\chi}] = \begin{cases} 1 & \text{if } \xi = \chi, \\ 0 & \text{if } \xi \neq \chi. \end{cases}$$

or equivalently,

$$E[\chi] = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

Since $\hat{R}$ is a set of N orthogonal functions in $\mathbb{C}^{|R|}$, they form a basis of $\mathbb{C}^{|R|}$, i.e., for every function $f : R \to \mathbb{C}$ there exist complex numbers α_χ for every $\chi \in \hat{R}$ such that

$$f = \sum_{\chi \in \hat{R}} \alpha_\chi \chi.$$

The coefficients α_χ are called the *fourier coefficients* of f and are typically written $\hat{f}(\chi) := \alpha_\chi$. In particular, $\hat{f}(\chi_0)$ is called the *principal* fourier coefficient and all other coefficients are referred as non-principal. Thus,

$$f = \sum_{\chi \in \hat{R}} \hat{f}(\chi)\chi$$

for any $f : R \to \mathbb{C}$. One has

$$\hat{f}(\chi) = E[f\overline{\chi}].$$

More precisely, this can be verified from the fact that

$$E[f\overline{\chi}] = E\left[\left(\sum_{\xi \in \hat{R}} \alpha_\xi \xi\right)\overline{\chi}\right] = E[\alpha_\chi \chi\overline{\chi}] = \alpha_\chi$$

using orthogonality. For any $f, g : R \to \mathbb{C}$, we have

$$E[fg] = E\left[\left(\sum_{\chi \in \hat{R}} \hat{f}(\chi)\chi\right)\left(\sum_{\xi \in \hat{R}} \hat{g}(\xi)\xi\right)\right] = \sum_{\chi, \xi \in \hat{R}} \hat{f}(\chi)\hat{g}(\xi)E[\chi\xi] = \sum_{\chi \in \hat{R}} \hat{f}(\chi)\hat{g}(\overline{\chi}).$$

and similarly $E[f\overline{g}] = \sum_{\chi \in \hat{R}} \hat{f}(\chi)\overline{\hat{g}(\chi)}$. In particular $E[|f|^2] = \sum_{\chi \in \hat{R}} |\hat{f}(\chi)|^2$ and if $f : R \to \{-1, 1\}$ then

$$\sum_{\chi \in \hat{R}} \hat{f}(\chi)^2 = 1$$

since $E[f^2] = 1$. Moreover if $f : R \to \{0, 1\}$ then $(-1)^f : R \to \{-1, 1\}$ and $(-1)^f = 1 - 2f$ so

$$\begin{aligned} 1 &= \sum_{\chi \in \hat{R}} \widehat{(-1)^f}(\chi)^2 \\ &= \sum_{\chi \in \hat{R}} \widehat{1 - 2f}(\chi)^2 \\ &= \sum_{\chi \in \hat{R}} (\hat{1}(\chi) - 2\hat{f}(\chi))^2 \\ &= \sum_{\chi \in \hat{R}} \hat{1}(\chi)^2 - 4\hat{1}(\chi)\hat{f}(\chi) + 4\hat{f}(\chi)^2 \\ &= 1 - 4\hat{f}(\chi_0) + 4\sum_{\chi \in \hat{R}} \hat{f}(\chi)^2 \end{aligned}$$

from which we deduce:

$$\hat{f}(\chi_0) = \sum_{\chi\in\hat{R}} \hat{f}(\chi)^2, \qquad (\textbf{whenever } f : R \to \{0,1\}). \tag{2}$$

Define convolution of $f_1, f_2 : R \to \mathbb{C}$ as

$$(f_1 * f_2)(x) = \sum_{y\in R} f_1(y)g(x-y) = NE_y[f(y)g(x-y)].$$

Using the fact that $\chi(x-y) = \chi(x)\overline{\chi(y)}$ for all $\chi \in \hat{R}$, x, y we find

$$\begin{aligned}
\widehat{f_1 * f_2}(\chi) &= E_x\left[(f_1*f_2)(x)\overline{\chi(x)}\right] \\
&= E_x\left[\sum_y f_1(y)f_2(x-y)\overline{\chi(x)}\right] \\
&= \frac{1}{N}\sum_y f_1(y)\sum_x f_2(x-y)\overline{\chi(x)} \\
&= \frac{1}{N}\sum_y f_1(y)\sum_x f_2(x)\overline{\chi(x+y)} \\
&= N\left(\frac{1}{N}\sum_y f_1(y)\overline{\chi(y)}\right)\left(\frac{1}{N}\sum_x f_2(x)\overline{\chi(x)}\right) \\
&= N\hat{f}_1(\chi)\hat{f}_2(\chi).
\end{aligned} \tag{3}$$

In fact, by virtue of associativity one may define a convolution $f_{(1*k)} := f_1 * f_2 * \cdots * f_k$ of any $f_1, f_2, \ldots, f_k$ and for any $k \geq 2$, in which case (3) has a natural generalization, namely

$$\hat{f}_{(1*k)}(\chi) = N^{k-1}\hat{f}_1(\chi)\hat{f}_2(\chi)\ldots\hat{f}_k(\chi). \tag{4}$$

For any (multi)set Z with elements from R, define $1_Z : R \to \mathbb{C}$ by the mapping

$$x \longmapsto |\{y \in Z \;:\; y = x\}|,$$

i.e., $1_Z(x)$ denotes the multiplicity of x in Z. Then, using (4), for any sets $B_1, B_2, \ldots, B_k \subseteq R$, we have

$$\begin{aligned}
\mu(A, B_1, B_2, \ldots, B_k) &= \sum_{x\in R} 1_A(x)1_{B_{(1*k)}}(x) \\
&= NE[1_A 1_{B_{(1*k)}}] \\
&= N\sum_{\chi\in\hat{R}} \hat{1}_A(\chi)\hat{1}_{B_{(1*k)}}(\overline{\chi}) \\
&= N^k \sum_{\chi\in\hat{R}} \hat{1}_A(\chi)\hat{1}_{B_1}(\overline{\chi})\hat{1}_{B_2}(\overline{\chi})\ldots\hat{1}_{B_k}(\overline{\chi}) \\
&= N^k\left(\frac{|A||B_1||B_2|\ldots|B_k|}{N^{k+1}} + \sum_{\chi\neq\chi_0} \hat{1}_A(\chi)\hat{1}_{B_1}(\overline{\chi})\hat{1}_{B_2}(\overline{\chi})\ldots\hat{1}_{B_k}(\overline{\chi})\right),
\end{aligned}$$

and, by rearranging terms

$$\mu(A, B_1, B_2, \ldots, B_k) - \frac{|A||B_1||B_2|\ldots|B_k|}{N} = N^k \sum_{\chi\neq\chi_0} \hat{1}_A(\chi)\hat{1}_{B_1}(\overline{\chi})\hat{1}_{B_2}(\overline{\chi})\ldots\hat{1}_{B_k}(\overline{\chi}).$$

It follows that

$$\left|\mu(A, B_1, B_2, \ldots, B_k) - \frac{|A||B_1||B_2|\ldots|B_k|}{N}\right| \leq N^k \sum_{\chi\neq\chi_0} |\hat{1}_A(\chi)||\hat{1}_{B_1}(\overline{\chi})||\hat{1}_{B_2}(\overline{\chi})|\ldots|\hat{1}_{B_k}(\overline{\chi})|.$$

Define $|\hat{1}_A| := \max_{\chi \neq \chi_0} |\hat{1}_A(\chi)|$. Then, letting $B_{>2} = B_3 \times \cdots \times B_k$, we have

$$\begin{aligned}\left|\mu(A,B_1,B_2,\dots,B_k) - \frac{|A||B_1||B_2|\dots|B_k|}{N}\right| &\leq N^k \cdot |\hat{1}_A| \sum_{\chi \neq \chi_0} |\hat{1}_{B_1}(\chi)||\hat{1}_{B_2}(\chi)|\dots|\hat{1}_{B_k}(\chi)| \\ &\leq N^2 \cdot |\hat{1}_A| \cdot |B_{>2}| \cdot \sum_{\chi \in \hat{R}} |\hat{1}_{B_1}(\chi)||\hat{1}_{B_2}(\chi)|,\end{aligned}$$

where the second inequality follows from the fact that $|\hat{1}_X(\chi)| \leq |\hat{1}_X(\chi_0)| = |X|/N$ for any $X \subseteq R$ and any $\chi \neq \chi_0$. By Cauchy-Schwarz inequality and (2), we have

$$\begin{aligned}\left|\mu(A,B_1,B_2,\dots,B_k) - \frac{|A||B_1||B_2|\dots|B_k|}{N}\right| &\leq N^2 \cdot |\hat{1}_A| \cdot |B_{>2}| \cdot \sqrt{\sum_{\chi \in \hat{R}} \hat{1}_{B_1}(\chi)^2} \sqrt{\sum_{\chi \in \hat{R}} \hat{1}_{B_2}(\chi)^2} \\ &= N^2 \cdot |\hat{1}_A| \cdot |B_{>2}| \cdot \sqrt{\hat{1}_{B_1}(\chi_0)} \sqrt{\hat{1}_{B_2}(\chi_0)} \\ &\leq N \cdot |\hat{1}_A| \cdot |B_{>2}| \cdot \sqrt{|B_1||B_2|} \qquad (5)\end{aligned}$$

Then, for all sets $B_1, B_2, \dots, B_k \subseteq R$, $|B_1| = |B_2| = \cdots = |B_k| = p$, we have

$$\left|\mu(A,B_1,B_2,\dots,B_k) - \frac{|A||B_1||B_2|\dots|B_k|}{N}\right| \leq p^{k-1} \cdot N \cdot |\hat{1}_A|. \qquad (6)$$

All that remains is to show that $N \cdot |\hat{1}_A| \in O(\ln(N)q)$ with overwhelmingly high probability. At this point the proofs for Theorem 2 and 3 diverge depending upon the tail inequality in play.

2.1 Proof of Theorem 2

This case adheres to the well-known Chernoff bound, as also observed previously in [Bab02, Ste13, CS18]. In particular, for any $\chi \neq \chi_0$ and an arbitrary ordering $(A_1, \dots, A_q)$ of A, we have

$$\begin{aligned}N \cdot |\hat{1}_A(\chi)| &= \left|\sum_x 1_A(x)\chi(x)\right| \\ &= \left|\sum_x \sum_{i=1}^q 1_{\{A_i\}}(x)\chi(x)\right| \\ &= \left|\sum_{i=1}^q \chi(A_i)\right|.\end{aligned}$$

Writing $\chi(A_i) = \phi(A_i) + \iota\psi(A_i)$ and splitting the corresponding sums, we have

$$\begin{aligned}N \cdot |\hat{1}_A(\chi)| &= \left|\sum_{i=1}^q \chi(A_i)\right| \\ &= \left|\sum_{i=1}^q \phi(A_i) + \iota \sum_{i=1}^q \psi(A_i)\right|,\end{aligned}$$

where $\phi(A_i), \psi(A_i)$ are real-valued random variables with $|\phi(A_i)|, |\psi(A_i)| \leq 1$ and $E_{A_i}[\phi(A_i)] = E_{A_i}[\psi(A_i)] = 0$. Furthermore, $\phi(A_i)$ are all independent, and similarly $\psi(A_i)$ are all independent. Then, for any $a \geq 0$, we have

$$\begin{aligned}\Pr\left(N \cdot |\hat{1}_A(\chi)| \geq a\right) &\leq \Pr\left(\left|\sum_{i=1}^q \phi(A_i)\right| \geq a\right) + \Pr\left(\left|\sum_{i=1}^q \psi(A_i)\right| \geq a\right) \\ &\leq 4e^{-a^2/2q},\end{aligned}$$

where the second inequality is a consequence of Chernoff bound. Finally, union bound gives

$$\Pr(N \cdot |\hat{1}_A| \geq a) \leq \sum_{\chi \neq \chi_0} \Pr(N \cdot |\hat{1}_A(\chi)| \geq a) \leq 4(N-1)e^{-a^2/2q}. \tag{7}$$

By setting $a = \sqrt{2(1+\epsilon)\ln(N)q}$ for $\epsilon > 0$

$$\left| \mu(A, B_1, B_2, \ldots, B_k) - \frac{qp^k}{N} \right| \leq p^{k-1}\sqrt{2(1+\epsilon)\ln(N)q}, \tag{8}$$

for all sets $B_1, B_2, \ldots, B_k \subseteq R$, $|B_1| = \cdots = |B_k| = p$ with at least $1 - 4/N^{\epsilon}$ probability.

2.2 Proof of Theorem 3

Hayes [Hay03] proved the following result.

Theorem 4 (Hayes, [Hay03] Lemma 6.3). *Let R be a finite abelian group of order N, and let χ be a non-principal character of R. Let $q \leq N$ and $q' = \min\{q, N-q\}$. For any $a > 0$, any without replacement sample $A = (A_i)_{1 \leq i \leq q}$ we have*

$$\Pr\left(N \cdot |\hat{1}_A(\chi)| \geq a\sqrt{q'}\right) \leq 2e^2 e^{-a^2/8}.$$

Then, the result follows by using $q \leq N/2$ and choosing $a = 2\sqrt{2(1+\epsilon)\ln(N)}$.

References

[Bab02] László Babai. The fourier transform and equations over finite abelian groups: An introduction to the method of trigonometric sums. Online Lecture Notes (Version 1.3), 2002. http://people.cs.uchicago.edu/ laci/reu02/fourier.pdf (last accessed: 7th March, 2024).

[CS18] Benoît Cogliati and Yannick Seurin. Analysis of the single-permutation encrypted davies-meyer construction. *Des. Codes Cryptogr.*, 86(12):2703–2723, 2018.

[Hay03] Thomas P. Hayes. A large-deviation inequality for vector-valued martingales. Online, 2003. https://www.cs.unm.edu/ hayes/papers/VectorAzuma/VectorAzuma20030207.pdf (last accessed: 7th March, 2024).

[Ste13] John P. Steinberger. Counting solutions to additive equations in random sets. *CoRR*, abs/1309.5582, 2013. http://arxiv.org/abs/1309.5582 (last accessed: 7th March, 2024).